Temperature fluctuations over a heated horizontal surface

By A. A. TOWNSEND
Emmanuel College, Cambridge

(Received 21 July 1958)

The previous work of Thomas (1956) on the turbulent convection over a single, heated, horizontal surface has been extended using improved methods of analysis of the temperature fluctuations, and it has been possible to measure the distributions of mean temperature, the mean squares of the temperature fluctuation, the temperature gradient and the rate-of-change of temperature, and the statistical distributions of these quantities. These measurements were made for three different values of the convective heat flux, and the results are consistent with the dimensional consequences of assuming that the convection near the surface is independent of the distant boundaries and determined by the heat flux and the viscosity and conductivity of the fluid.

The most striking feature of the observations is that the fluctuations of temperature, temperature gradient, and rate-of-change of temperature, all show periods of activity, characterized by large fluctuations, alternating with periods of quiescence with comparatively small ones. Both the proportion and frequency of occurrence of the active periods decrease with increasing distance from the surface and they probably occur when rising columns of hot air pass through the point of observation. The quiescent periods occur when the point of observation lies outside the columns, and analysis of the statistical distributions of the various fluctuations shows that, during these periods, they are nearly independent of height. It is concluded that the quiescent fluctuations are typical of the turbulent convection far from the surface while the active fluctuations are the manifestation of the convective processes arising near the rigid boundary. These processes may be described as the detachment of columns of hot air from the edge of the conduction layer and the erosion of these rising columns by contact with the surrounding air which is in vigorous turbulent motion. Since the variations of the intensities with height is dominated by the contributions of the active periods, it is not surprising that no agreement is found with the predictions of the similarity theory which assumes the convection to be independent of the conduction layer at a sufficient distance from it. The Malkus theory of turbulence, which emphasizes dependence on the conduction layer, is in qualitative agreement with this inferred mechanism of the convection and is in quantitative agreement with the observed distribution of mean temperature. A brief discussion is given of the effect of a horizontal shearing motion on the convection and of the relation of these measurements to measurements of the temperature distribution in the earth's boundary layer with upward flux of total heat.

1. Introduction

Although the convective motion over a heated, horizontal surface is a simple form of turbulence with desirable properties of homogeneity and symmetry, there have been very few investigations of the fluctuations in velocity and temperature that underlie the gross phenomenon of heat transport. Malkus (1954*a*) was able to estimate turbulent velocities by observing the motion of suspended particles and Thomas (Thomas & Townsend, 1957) measured intensities and autocorrelation functions of the temperature fluctuations, but the experimental material is very meagre by comparison with our knowledge of the turbulent motion in shear flow and it was considered useful to continue and extend the earlier measurements of Thomas. Another, less general, reason for this investigation was a desire to test as fully as possible the predictions of a novel theory of turbulence proposed by Malkus (1954*b*, 1956), which seeks to relate the transport properties of the fully turbulent flows to the neutrally stable disturbances of the corresponding laminar flows and which, without introducing disposable constants, has succeeded in predicting the mean velocity distribution for turbulent flow in a two-dimensional channel with reasonable accuracy. Although the absence of arbitrary constants makes the theory attractive, its basic assumptions are neither immediately convincing nor open to direct experimental verification and the possibility of deriving similar results using established principles of similarity is an obstacle to its acceptance. However, applied to convection between parallel horizontal planes, the Malkus theory predicts a distribution of mean temperature quite different from that predicted by arguments of similarity and that the convection is everywhere dependent on the viscosity and conductivity of the fluid. Both the massive evidence that turbulent transport processes become independent of viscosity and conductivity for large Reynolds numbers of flow and the details of the theory suggest that, if this happens, it is because the whole convection depends on processes occurring close to the surface where thermal conductivity dominates the heat transfer and viscous stresses the turbulent motion. So the Malkus theory may be tested in two ways, by comparison of the observed and predicted variations of mean temperature with height and by looking for evidence of an intimate connexion between local temperature and processes occurring near the surface. On the other hand, the similarity theory, which successfully describes many aspects of turbulent flow in channels, rules out dependence on processes occurring in the conduction layer and predicts definite functional forms for the variations of mean temperature, mean square temperature fluctuations, and a number of other quantities.

The experiments to be described form a detailed study of the temperature fluctuations in a practical system of natural convection that approximates to the ideal one considered in the theory. The particular characteristics of the fluctuations that were measured are simply those that could be measured conveniently without excessive complication of equipment or observational difficulty.

2. Notation

The ideal system, to which the experimental arrangement is an approximation, is convection over a single, infinite, horizontal surface in a perfect gas without other boundaries, and the motion and temperature field may be described using a system of rectangular co-ordinates in which the Oz axis is vertical and the origin is in the surface. The convection should be statistically homogeneous with respect to variation of the horizontal co-ordinates. The following notation will be used:

u, v, w	are the velocity components parallel to Ox, Oy and Oz respectively,
$q^2 =$	$u^2+v^2+w^2$,
P, p	are the mean pressure and the pressure fluctuation from the mean,
T, θ	are the mean absolute temperature and the temperature fluctuation from the mean,
ν	is the kinematic viscosity,
k	is the thermal conductivity,
κ	is the thermometric conductivity,
ρ	is the fluid density,
H	is the constant upward flux of total heat,
$Q = \dfrac{H}{\rho c_p T}$	is a constant flux of buoyancy,
$z_0 = \left(\dfrac{\kappa^3}{gQ}\right)^{\frac{1}{4}}$	is a scale of length,
$u_0 = (\kappa g Q)^{\frac{1}{4}}$	is a scale of velocity,
$\theta_0 = \left(\dfrac{Q^3}{\kappa g}\right)^{\frac{1}{4}}$	is a scale of logarithmic temperature,
T_1	is the absolute temperature at the surface,
T_a	is the ambient or reference temperature, measured at a point 40 cm. above the surface,
N	is the frequency with which the instantaneous temperature passes through its mean value from below,
$S(x)$	is the probability that the instantaneous value of a fluctuating quantity shall exceed x,
$P(x) = -\dfrac{dS(x)}{dx}$	is the probability density function for the quantity.

It may be shown that z_0, u_0 and θ_0 are scales whose variation is sufficient to describe the change in the convection with different heat transfers, provided that the motion and temperature field close to the surface are unaffected by the distant boundaries and by the mechanism of removal of heat from the system and that fluids of only one Prandtl number are used. Measurements of heat transfer (e.g. Malkus, 1954*a*) and of mean temperature near the heated surface show that the variation of mean temperature is concentrated in a shallow surface layer and is effectively independent of the other boundaries. This might suggest that the detailed structure of this surface layer is also determined by the heat flux and the fluid properties, and then the only change produced by a change in heat flux is the alteration in the characteristic scales. How far this is true is a question that the present work may help to answer.

3. Experimental arrangements

The practical convection system is an open-topped box, with bottom a horizontal and uniformly heated plate, 30 cm × 40 cm whose construction has been described before (Thomas & Townsend 1957). The sides of the box are of hardboard 56 cm high, and are intended to ensure that convective transport of heat is as far as possible by turbulent transport and as little as possible by steady streaming motions. The boundary conditions of this system are defined by the surface temperature and the room temperature if we ignore unwanted but

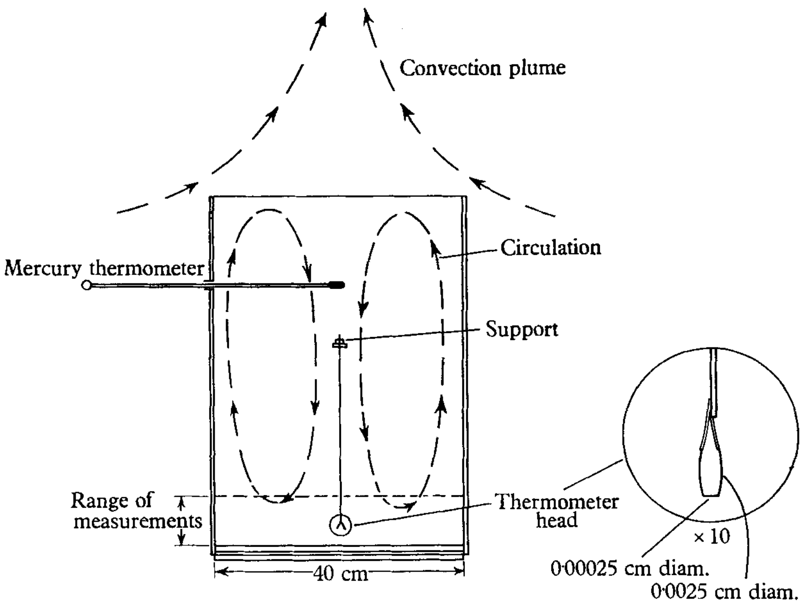

FIGURE 1. Diagrammatic section of the convection box. (The dashed lines indicate the form of the steady circulation inferred from the horizontal distribution of mean temperature.)

unavoidable disturbances from air-currents in the room, while the ideal system is defined by surface temperature and by the temperature at infinity. For the purpose of relating experimental measurements to the ideal system, it is better to use as 'temperature at infinity' the temperature at the base of the convection plume rising from the open top of the box rather than the room temperature. In these experiments, the temperature at a point 40 cm above the surface (16 cm below the open top) was measured with a mercury-in-glass thermometer and used as a reference temperature and as an approximation to the effective 'temperature at infinity'. During the course of a run it was not unusual for the room temperature and the reference temperature to change by one or two degrees and it was necessary to correct the observed mean temperatures for this drift. This was done by assuming that all temperatures in the system changed by the same amount and subtracting from any observed temperature the excess of room temperature over 20·0 °C. In all the experiments the room temperature lay between 17° and 23 °C.

With this arrangement the most likely causes of deviations from ideal behaviour are the presence and finite height of the side-walls of the box. The simple presence of side-walls may make itself felt in two ways, by restrictions on horizontal movement of the fluid and by steady circulations of large scale induced by unequal temperature gradients near the walls and in the centre of the box. It is not very likely that horizontal motion could be seriously restricted in the layer of depth 8 cm in which all measurements were made, but there is some evidence of a weak

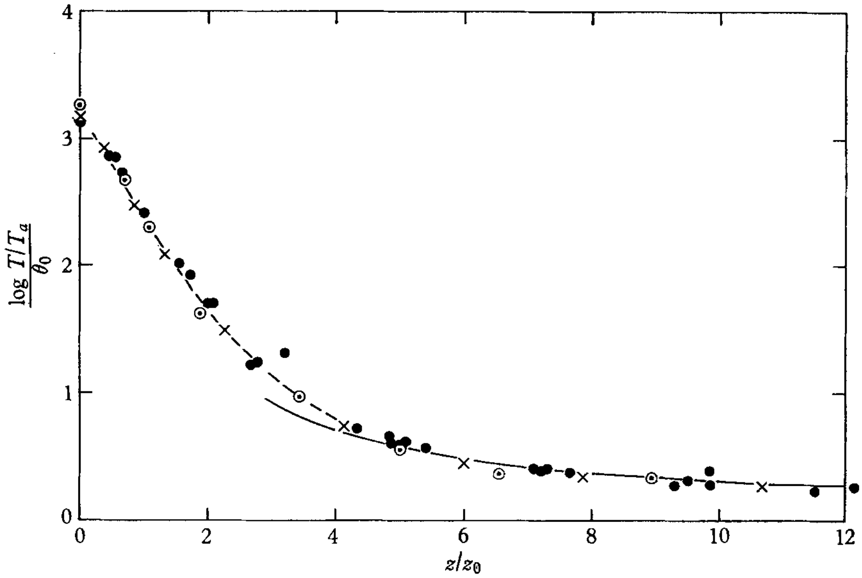

FIGURE 2. Distribution of mean temperature for z/z_0 less than 12. (•, heat flux A; ×, heat flux B; ⊙, heat flux C.)

circulation with cold air descending near the centre of the box and warm air rising near the edges (figure 1). Whether this circulation is due to the side-walls, or whether it is an eddy induced by the convection plume from the open top, is difficult to determine but it is believed to be of too low an intensity to be of much significance for the flow as a whole. It may be pointed out that the measurements of Thomas showed good agreement between the heat flux computed from the heat loss over the whole plate and the heat flux computed from the local temperature gradient near the middle of the plate, a result that is unlikely if the circulatory motion carries with it appreciable heat. The finite height of the side-walls might lead to two types of disturbance, transient disturbances by uncontrolled draughts in the room and a steady disturbance induced by the convection plume. The relative importance of these will depend on the absolute strength of the convection plume.

All the results of the measurements are presented in the non-dimensional form appropriate to the ideal system, with units the non-dimensional scales of length, velocity and logarithmic temperature that may be constructed from the buoyancy flux Q, the gravitational acceleration and the thermometric conductivity. Logarithms of the absolute temperature are used instead of ordinary temperature, a procedure which is known to lead to better correlation between convection experiments with widely different temperature loadings and which has some theoretical justification (Thomas & Townsend 1957). A general inspection of figures 2–10, which show the variation with height of a variety of mean values

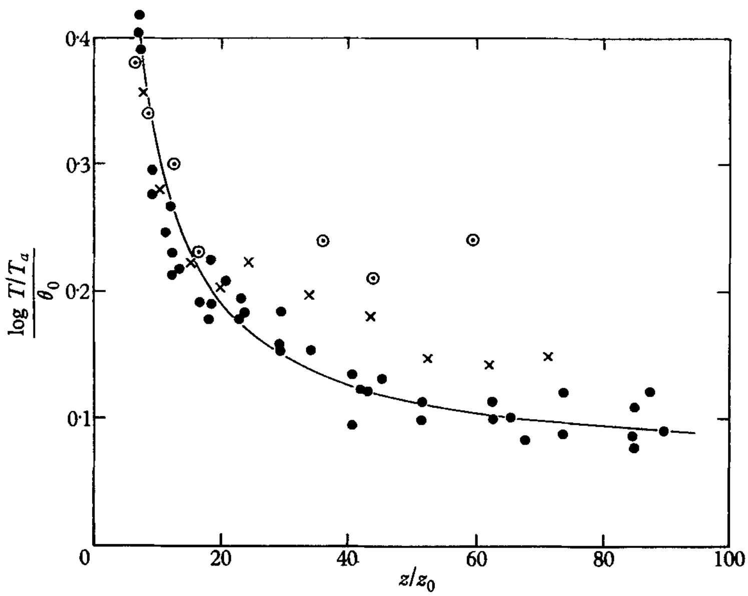

FIGURE 3. Distribution of mean temperature for z/z_0 greater than 8. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

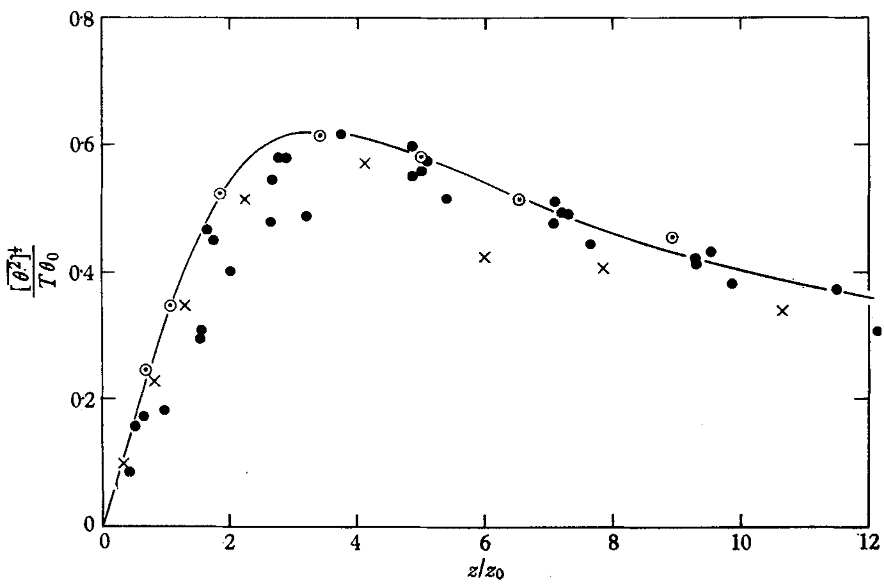

FIGURE 4. Distribution of root-mean-square temperature fluctuation for z/z_0 less than 12. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

connected with the temperature field for three different values of the heat flux, shows that the results of dimensional analysis of the ideal system are valid for the experimental system except within the conduction layer and at considerable distances from the surface. The discrepancies within the conduction layer (z/z_0

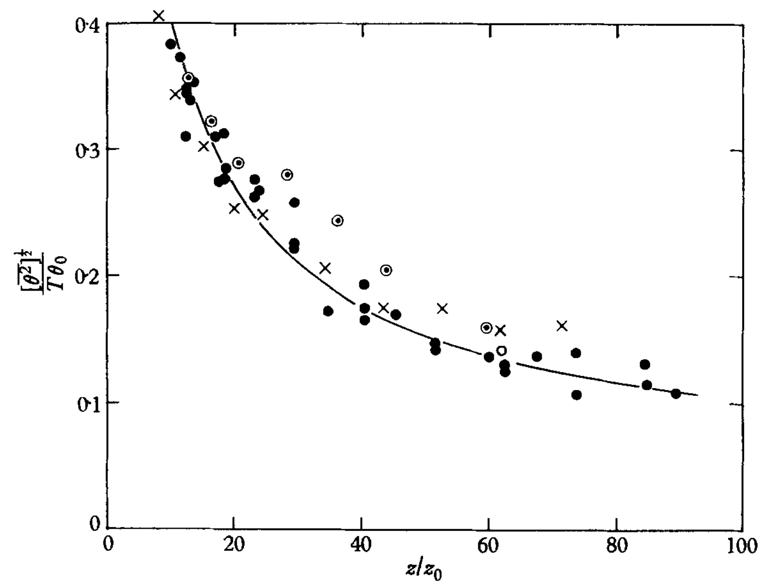

FIGURE 5. Distribution of root-mean-square temperature fluctuation for z/z_0 greater than 8. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

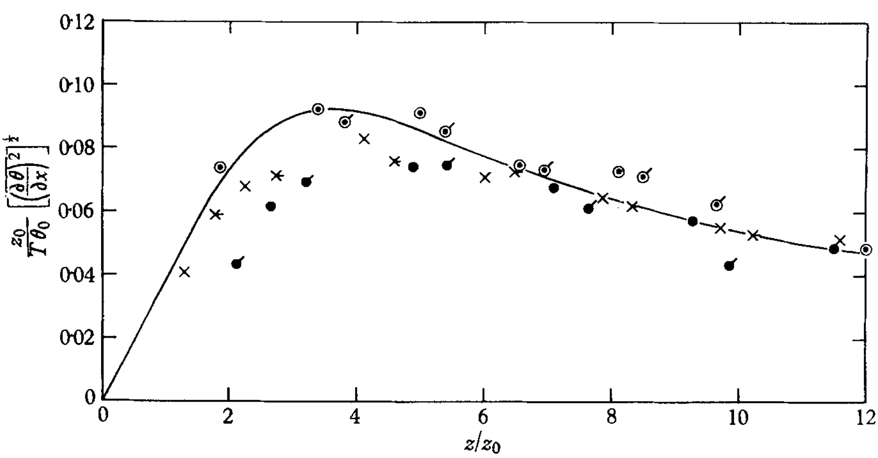

FIGURE 6. Distribution of root-mean-square temperature gradient for z/z_0 less than 12. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

less than three) are caused by the variation with temperature of thermometric conductivity and kinematic viscosity (the variations are most clearly distinguishable in the measurements of temperature fluctuations and temperature gradients in figures 4 and 6), which the dimensional analysis assumes to have the same values in all parts of the flow, an approximation nearly valid outside the conduction layer but not inside it for these surface temperatures. The discrepancies that become noticeable for large values of z/z_0 are probably due to deviations from ideal behaviour induced by the open top, in particular by air-currents in the room. Consistently with this conclusion, the measurements for

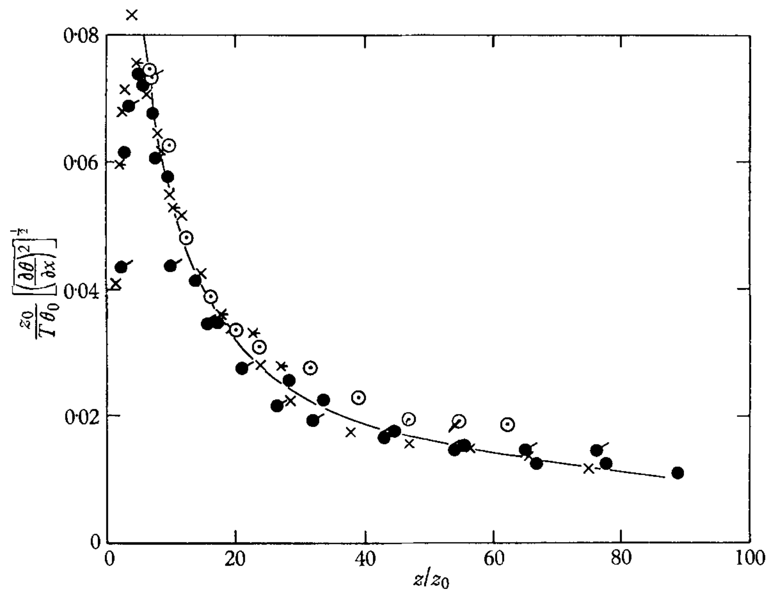

FIGURE 7. Distribution of root-mean-square temperature gradient for z/z_0 greater than 8. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

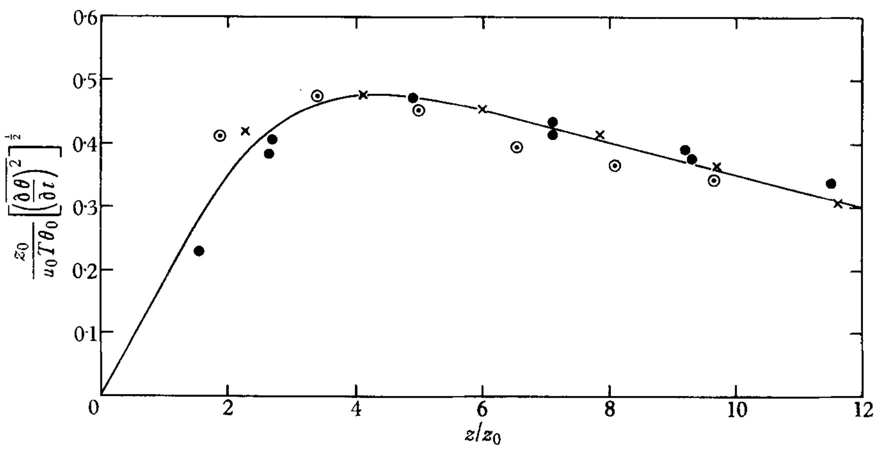

FIGURE 8. Distribution of root-mean-square rate of change of temperature for z/z_0 less than 12. (●, heat flux A; ×, heat flux B; ⊙, heat flux C.)

the smaller heat fluxes and the weaker convection plumes deviate from the measurements for the largest heat flux at heights that increase as the heat flux increases. For the lowest heat flux, deviations begin at $z/z_0 = 20$ (the exact value depends on the quantity measured) but for the intermediate flux deviations begin at $z/z_0 = 40$. Below these values no systematic departures from dimensional similarity are to be found, and it is reasonable to suppose that the observations for the largest heat flux represent the ideal system sufficiently well for z/z_0 less than sixty. Within the conduction layer, the ideal behaviour is best represented by observations at the lowest heat flux and the curves have been drawn with this in mind.

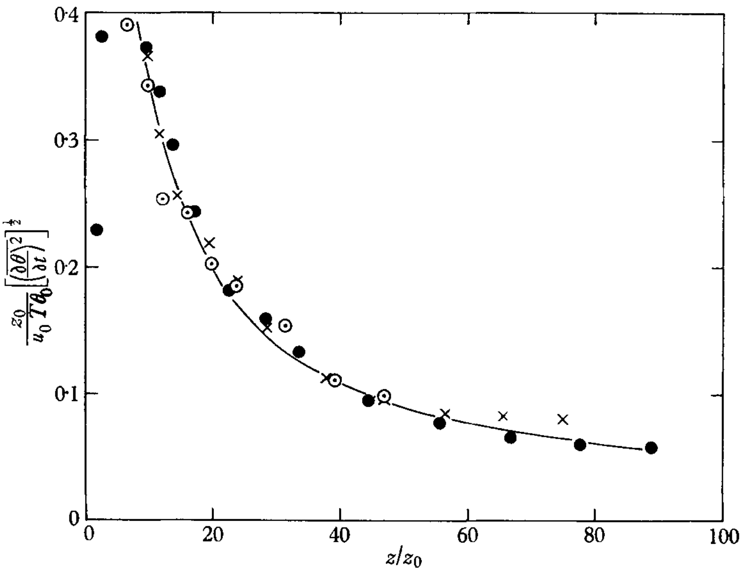

FIGURE 9. Distribution of root-mean-square rate of change of temperature for z/z_0 greater than 8. ($\bullet$, heat flux A; $\times$, heat flux B; $\odot$, heat flux C.)

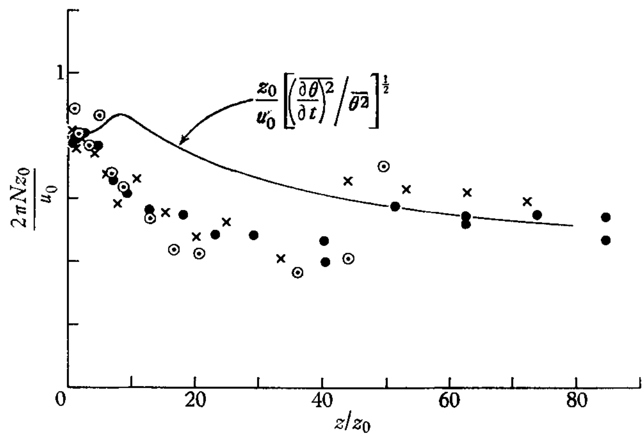

FIGURE 10. Distribution of mean frequency of zero temperature fluctuation. (Full line represents theoretical values if θ and $\partial\theta/\partial t$ are statistically independent and normally distributed.)

4. Use of a resistance thermometer to measure temperature fluctuations

The measurement of temperature in a turbulent motion with small or zero mean velocity presents the special problem of interference with the motion by the thermometer element and its supports. In the earlier work, the thermometer element was supported by parallel wires stretched horizontally across the convection box, an arrangement convenient for measurements between parallel planes. In this series of measurements, the flow is open from the top and the element is carried by a vertical length of hypodermic tubing, 1 mm in diameter and 15 cm long, itself supported by a brass tube 3 mm in diameter and 15 cm long. The whole is carried by a bridge whose height above the surface can be adjusted by micrometer screws. The sensitive element is a short length (1–2 mm) of platinum wire, of diameter $2\cdot5\ \mu$, etched from Wollaston wire and supported by pieces of unetched wire of diameter $25\ \mu$ and length about 5 mm (figure 1). In still air the time constant of such a wire is about 0·5 msec, and there is no doubt that the resistance of the element responds to temperature fluctuations more rapidly than the electrical circuit can respond to the changes in resistance. The wire current was sufficiently low to avoid appreciable ohmic heating.

The question is, how closely does the temperature measured by this arrangement resemble the temperature in the absence of the thermometer element and its support? Differences could arise in two ways, from the influence of the thermometer on the whole motion and from inability of the thermometer to measure temperature accurately, especially if there is appreciable conduction of heat along the supports or if the approach of fluid from certain directions is impeded by the supports. The effect of the thermometer assembly on the whole motion is difficult to assess, and the reasons for considering it to be negligible are that these measurements of mean temperature agree well with earlier measurements using the horizontal support and that measurements with a variety of thermometer heads are internally consistent. For similar reasons, heat conduction by the supports and support interference are believed to cause errors less than the statistical uncertainty of the measurements. Radiative exchange of heat with the surroundings has a negligible effect on the equilibrium temperature of wires of diameter $2\cdot5\ \mu$.

In order to obtain useful approximations to the mean values, it was necessary to take averages over periods of 10 min, and this was conveniently done by converting the electrical output of the resistance thermometer into pulses whose repetition rate was a linear function of the wire resistance. The electrical circuits used to do this and to make the other measurements are described in another paper (Townsend 1959).

5. Measurement of defining parameters

The ideal convection flow considered in the theory may be specified either by the flux of total heat or by the temperatures at the heated surface and at infinity. The definition of 'temperature at infinity' in the experimental arrangement has been discussed already, and the reference temperature is considered to be a good approximation to it. The surface temperature was measured in two ways, by thermocouples embedded in the dural plate and by extrapolation of the measurements of mean temperature to the position of the surface. This position can be established with considerable accuracy from measurements of the root-mean-square of the temperature fluctuation, a quantity that is proportional to distance from the surface within the conduction layer (figure 4).

The convective flux of total heat may be measured directly by subtracting from the total heat loss of the surface the computed loss by radiation, but the earlier work showed that these values were in good agreement with values computed from the observed gradient of mean temperature near the surface using the relation

$$H = -k\left[\frac{\partial T}{\partial z}\right]_{z=0}, \tag{5.1}$$

and that there was little to choose between the apparent accuracy of each set of values. In view of the uncertainty in the radiative heat loss, it was decided to use the second method which has the advantage of measuring all the parameters with one basic instrument, the platinum wire resistance thermometer.

Run	T_1 (°C)	T_a (°C)	$H \times 10^2$ (W. cm^{-2})	$\log \dfrac{T_1}{T_a}$	Q (cm sec^{-1})	θ_0	z_0 (cm.)	u_0 (cm sec^{-1})	$\theta_0^{-1} \log \dfrac{T_1}{T_a}$
B*	93·8	27·0	5·60	0·2011	0·157	0·0658	0·0887	2·39	3·06
C*	86·6	27·0	4·95	0·1817	0·139	0·0600	0·0914	2·32	3·03
F 2†	94·0	27·7	5·26	0·1995	0·148	0·0628	0·0903	2·36	3·18
F 5	91·0	27·0	5·15	0·1932	0·145	0·0618	0·0905	2·34	3·13
F 1†	63·1	23·9	2·78	0·1240	$0{\cdot}078_4$	0·0392	0·1044	2·00	3·16
F 4	60·3	23·9	2·54	0·1154	$0{\cdot}071_4$	0·0365	0·107	1·95	3·16
F 3	42·8	22·2	1·85	0·0674	0·0334	0·0207	0·128	1·61	3·26

* These runs were made with light cardboard walls of height 35 cm.

† These runs were made with side-walls of height 28 cm.

Note. All measurements have been corrected for deviations of room temperature from 20 °C.

TABLE 1

Heat flux	θ_0	z_0 (cm)	u_0 (cm sec^{-1})	$T_1 - 20$ (°C)	Q (cm sec^{-1})
A	0·0618	0·0905	2·34	71·0	0·145
B	0·0365	0·107	1·95	43·1	0·0714
C	0·0207	0·128	1·61	22·8	0·0334

TABLE 2

Table 1 shows the results of these measurements for seven distinct runs with various surface temperatures and experimental arrangements, together with the computed values of the buoyancy flux Q, of the characteristic scales of logarithmic temperature, length and velocity, and of the heat transfer coefficient, $\log T_1/T_a\,(kg/Q^3)^{\frac{1}{4}}$. It will be noticed that the measured values of the heat transfer coefficient have a mean value of $3{\cdot}14 \pm 0{\cdot}03$ and a standard deviation of 2·3 %, and that no correlation with heat flux or wall height is visible. This result is further confirmation of the hypothesis that the convective heat loss is determined by the motion and structure of a thin surface layer, too thin to be affected by external influences peculiar to the particular experimental arrangement.

The runs, F 3, F 4, F 5, were made with standard values of the electrical power input to the plate assembly, and the values of the scales computed from these runs were used in the reduction of later measurements with the same power inputs and, presumably, the same heat fluxes. In some of the later runs temperature measurements were made in the conduction layer and the assumed values of the convective heat loss were confirmed. For convenience, the characteristic scales for the three standard heat fluxes, which will be distinguished by the letters, A, B, C, are collected in table 2.

6. Mean temperatures and intensities of temperature fluctuations

Although the measurement of mean temperature was not the first object of the work, a large number of measurements were produced in the course of measuring mean squares of the temperature fluctuation and some of these are shown in figures 2 and 3. Within the limitations already mentioned, the consistency of measurements for different heat fluxes is good and the experimental points lie around a common curve that is consistent with the earlier measurements although the range of z/z_0 is much greater. The scatter of the observations is such that they might be represented for z/z_0 greater than ten by a power law,

$$\theta_0^{-1}\log\frac{T}{T_a} = A\left(\frac{z}{z_0}\right)^{-n} + B, \tag{6.1}$$

with almost any value of n between 0·3 and 1·5. If we select $n = 1$, corresponding to the Malkus theory, the best representation is

$$\theta_0^{-1}\log\frac{T}{T_a} = 2{\cdot}6\left(\frac{z}{z_0}\right)^{-1} + 0{\cdot}06 \tag{6.2}$$

for z/z_0 greater than 8, although an extreme possibility is

$$\theta_0^{-1}\log\frac{T}{T_a} = 2{\cdot}0\left(\frac{z}{z_0}\right)^{-1} + 0{\cdot}07 \tag{6.3}$$

for z/z_0 greater than 10. The additive constant in each of these expressions corresponds with an effective ambient temperature (or 'temperature at infinity') somewhat above the reference temperature. If we select $n = \frac{1}{3}$, as predicted by the similarity theory, the best representation is

$$\theta_0^{-1}\log\frac{T}{T_a} = 0{\cdot}95\left(\frac{z}{z_0}\right)^{-\frac{1}{3}} - 0{\cdot}12 \tag{6.4}$$

for z/z_0 greater than 8. Representation of the results by (6.4) is decidedly less satisfactory than representation by (6.2), and the multiplicative constant is very different from the one inferred from the measurements of Swinbank (Priestley 1954, 1955, 1956) which indicate that

$$\theta_0^{-1}\log\frac{T}{T_a} = 3{\cdot}5\left(\frac{z}{z_0}\right)^{-\frac{1}{3}} + \text{constant}. \tag{6.5}$$

It is only expected that the variation of mean temperature should be described by a power law if nearly all the heat flux is carried by turbulent movements, i.e. if $\overline{\theta w} \gg -\kappa(\partial T/\partial z)$. The ratio of the two sides of this inequality is easily computed using any of the possible representations, and the condition is strongly satisfied for z/z_0 greater than 8 ($-\kappa[\partial T/\partial z]/\overline{\theta w} = 0{\cdot}04$ for $z/z_0 = 8$).

Mean squares of the temperature fluctuations were measured at the same time, with the results shown in figures 4 and 5. For z/z_0 less than 1·5, the root-mean-square of the fluctuation is nearly proportional to z/z_0, although the constant of proportionality depends on the temperature loading, $\log T_1/T_a$. For the lowest heat flux C, which corresponds most nearly with the condition of very small temperature loading assumed in the basic theory,

$$\frac{(\overline{\theta^2})^{\frac{1}{2}}}{T\theta_0} = 0{\cdot}35\frac{z}{z_0} \quad (z/z_0 < 1{\cdot}5). \tag{6.6}$$

The fluctuation intensity rises to a maximum value near $z/z_0 = 3$ and then decreases in a manner adequately described by

$$\frac{(\overline{\theta^2})^{\frac{1}{2}}}{T\theta_0} = 1{\cdot}60\left(\frac{z}{z_0}\right)^{-0{\cdot}6} \tag{6.7}$$

for z/z_0 greater than 6.* It is not possible to represent the measurements by the power-law predicted by the similarity theory

$$\frac{(\overline{\theta^2})^{\frac{1}{2}}}{T\theta_0} \propto \left(\frac{z}{z_0}\right)^{-\frac{1}{3}}. \tag{6.8}$$

The points in figures 4 and 5 include observations with single and with double thermometer elements, and the close agreement shows that increased size and complexity of the thermometer has little effect on the measured fluctuations. It is therefore reasonable to believe that the resistance thermometer detects temperature fluctuations resembling closely those that would occur in the absence of the thermometer.

7. Gradients and rates-of-change of temperature

Temperature gradients were measured with a thermometer head with two sensitive elements, each about 1·5 mm long and held parallel to each other with a horizontal separation of 0·212 cm. If the elements are connected in adjacent arms of the Wheatstone bridge, the output of the bridge is a linear function of the resistance ratio and, if the lead resistance is negligible and the temperature fluctuations small compared with the absolute temperature, of the temperature difference. The mean square of the temperature difference is determined in the usual way and the mean square gradient is computed by dividing by the square of the separation. The extent to which the finite difference resembles the local gradient was investigated in a second series of measurements in which the separation of the elements was 0·461 cm. Since, for small separations,

$$\overline{\left(\frac{\theta(x)-\theta(x+r)}{r}\right)^2} = \overline{\left(\frac{\partial\theta}{\partial x}\right)^2} - \frac{r^2}{12}\overline{\left(\frac{\partial^2\theta}{\partial x^2}\right)^2}, \tag{7.1}$$

* No significance attaches to the exact value of the exponent which has been rounded off to the nearest 0·05. These and following power-laws (equations 7.3–4) merely represent in a compact form the experimental measurements over the range of observation and may have no validity outside this range.

a comparison of the two series of measurements enables the possible error to be determined. It was found to be rather less than the scatter of the observations and has been neglected. Figures 6 and 7 show the root-mean-squares of the temperature gradients obtained in this way. The variation with height is generally similar to that of the root-mean-square temperature fluctuation, although the effects of temperature loading at small values of z/z_0 are rather larger. For the lowest heat flux,

$$\frac{z_0}{T\theta_0}\left[\overline{\left(\frac{\partial\theta}{\partial x}\right)^2}\right]^{\frac{1}{2}} = 0{\cdot}4\frac{z}{z_0} \tag{7.2}$$

for z/z_0 less than 1·5, while for the largest values of z/z_0 a good representation is

$$\frac{z_0}{T\theta_0}\left[\overline{\left(\frac{\partial\theta}{\partial x}\right)^2}\right]^{\frac{1}{2}} = 0{\cdot}30\left(\frac{z}{z_0}\right)^{-0{\cdot}75} \quad (z/z_0 > 6). \tag{7.3}$$

The similarity theory predicts an exponent of $-\frac{2}{3}$.

Intensities of the time-rate of change of temperature were measured using an electrical differentiating circuit with the results shown in figures 8 and 9. For z/z_0 greater than 6, the measurements are described by

$$\frac{z_0}{u_0\theta_0 T}\left[\overline{\left(\frac{\partial\theta}{\partial t}\right)^2}\right]^{\frac{1}{2}} = 2{\cdot}48\left(\frac{z}{z_0}\right)^{-0{\cdot}85}. \tag{7.4}$$

The similarity theory predicts an exponent of $-\frac{1}{3}$.

A convenient method of obtaining information about the time-rate of change of a fluctuating quantity is to measure the frequency with which it passes through a particular value, usually the mean value (see, for example, Liepmann 1952). It may be shown (Rice 1944, 1945) that the frequency N with which the temperature fluctuation passes through its mean value from below is

$$N = \int_0^\infty P(0\,|\,\dot{\theta})\,\dot{\theta}\,d\dot{\theta}, \tag{7.5}$$

where $P(\theta\,|\,\dot{\theta})$ is the joint probability distribution function for the temperature θ and the time derivative of the temperature $\dot{\theta}$. If the temperature fluctuation and its rate of change are statistically independent,

$$N = \tfrac{1}{2}P(0)\,\overline{|\dot{\theta}|} \tag{7.6}$$

and, if both are normally distributed,

$$N = \frac{1}{2\pi}[\overline{\dot{\theta}^2}/\overline{\theta^2}]^{\frac{1}{2}}. \tag{7.7}$$

Measurements of this rate have been made and are compared with direct measurements of $[\overline{\dot{\theta}^2}/\overline{\theta^2}]^{\frac{1}{2}}$ in figure 10. Good agreement is found only at large and at small values of z/z_0, and it is inferred that substantial deviations from a normal distribution may exist.

8. Statistical distributions of temperature

Comparison of figures 3 and 5 will show that the root-mean-square temperature fluctuation is very roughly equal to the difference between the mean and reference temperatures for all values of z/z_0 greater than five, so it is not surprising that measurements of the statistical distribution of instantaneous temperature show that 'cold' air occurs with significant frequency in all parts of the flow outside

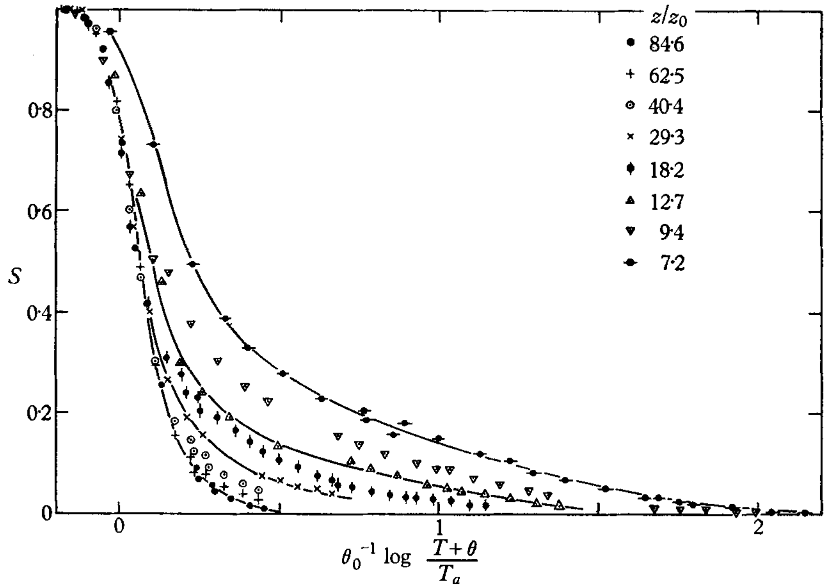

FIGURE 11. Statistical distributions of instantaneous temperature for large values of z/z_0.

the conduction layer. The way in which this happens is more surprising. The most obvious feature of figure 11, which shows the probabilities of occurrence of temperatures greater than set values, is that the low temperature parts of the curves are nearly identical in shape, particularly for the larger values of z/z_0. This behaviour suggests that $S(x)$, the probability that the temperature should exceed x, is of the form,

$$S(x) = (1-\gamma)\, S_1(x) + \gamma S_2(x), \tag{8.1}$$

where S_1 is independent of height and γ is a duration factor which decreases with increase in distance from the surface.* The interpretation of (8.1) is that the temperature at a point may vary in two distinct ways, either with small fluctuations and a low mean temperature or with large fluctuations and a high mean temperature. More definite evidence of the existence of two modes of fluctuation is to be found in measurements of the statistical distributions of temperature gradients and rates of change, but it is also strongly suggested by visual observations of the behaviour of the measuring equipment. Comparatively infrequent 'active' periods during which the temperature oscillates wildly between wide limits alternate with extended periods of quiescence during which the tempera-

* See figure 23, in which $S(1-\gamma)^{-1}$ is plotted as a function of temperature.

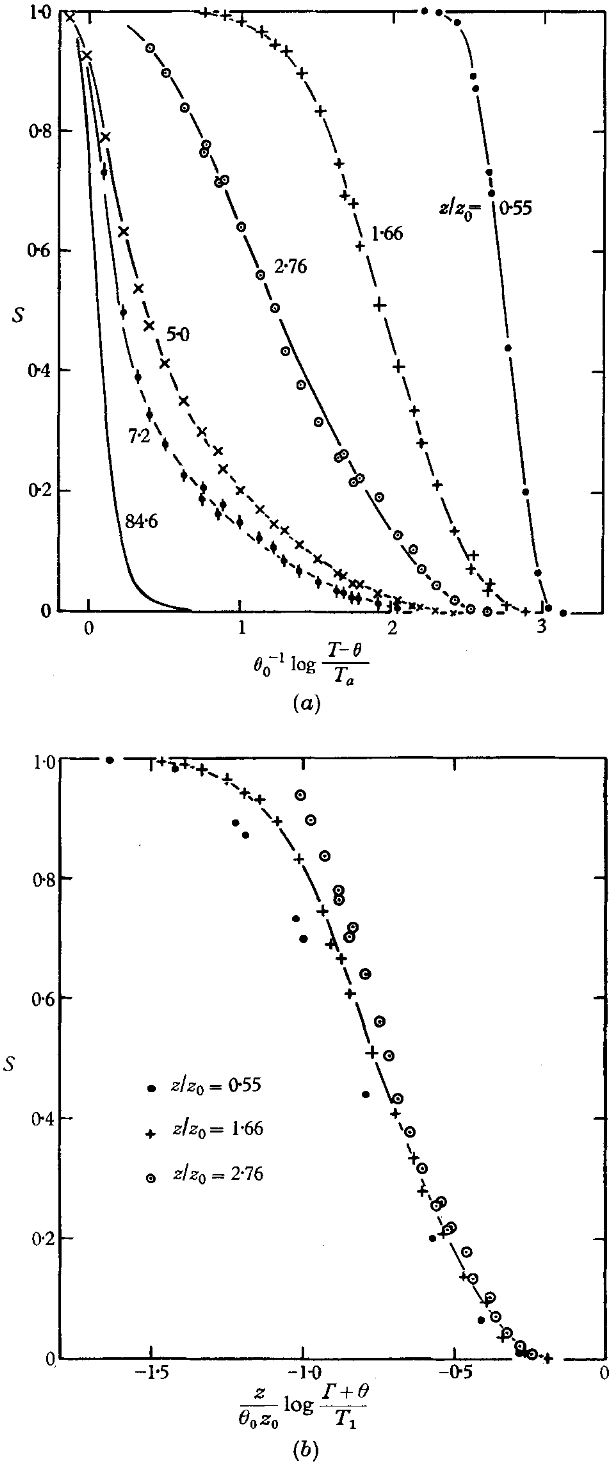

FIGURE 12. Statistical distributions of instantaneous (a and b) temperature for small values of z/z_0.

ture is low and varies very little. The relative duration and frequency of the active periods decreases with increasing height.

Within and near the conduction layer, it is not possible to distinguish two components of the fluctuation, and, for z/z_0 less than 2, the integrated probability function is nearly of the form

$$S(T+\theta) = S_w\left(\frac{z}{z_0}\theta_0^{-1}\log\frac{T+\theta}{T_1}\right), \tag{8.2}$$

as expected in a region dominated by the effects of conduction. Figure 12 shows distributions of temperature fluctuations in and near the conduction layer plotted against $\theta_0^{-1}\log[(T+\theta)/T_a]$ and $(z/z_0)\theta_0^{-1}\log[(T+\theta)/T_1]$. It may be noticed (i) that, for $z/z_0 = 2{\cdot}76$, $\theta_0^{-1}\log[(T+\theta)/T_a]$ is observed to assume all values between $-0{\cdot}1$ and $+2{\cdot}64$, which is 80% of the total range of temperature, and (ii) that the statistical similarity of temperature fluctuations within the conduction layer expressed by equation (8.2) is most persistent for the higher temperatures, i.e. the smaller temperature gradients.

9. Statistical distributions of the space and time derivatives of the temperature fluctuation

The statistical distributions of temperature gradient and rate of change of temperature suggest strongly the existence of two modes of fluctuation. Figures 13 and 14, which show on a Gaussian probability plot the distributions of temperature gradient, $\partial\theta/\partial x$, for $z/z_0 = 7{\cdot}1$ and $22{\cdot}6$, are naturally interpreted as composite distributions. The behaviour at large deviations will be determined by the

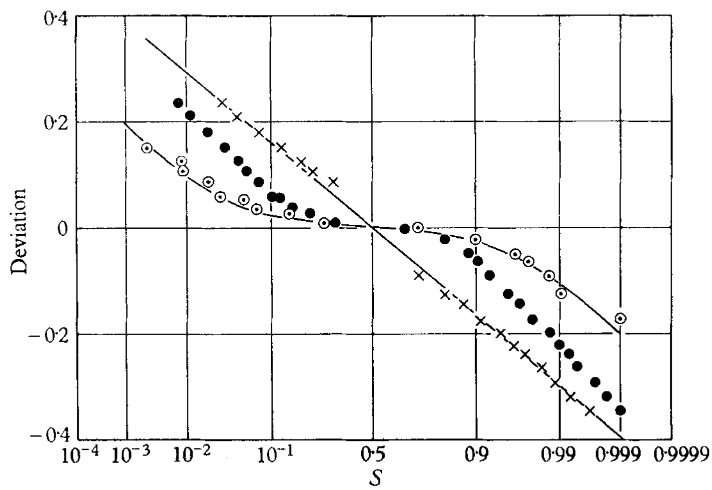

FIGURE 13. Statistical distribution of temperature gradient, $\partial\theta/\partial x$, at $z/z_0 = 7{\cdot}1$ on a Gaussian probability plot. ($\bullet$, original measurements; $\times$, $S/\gamma \to S_1$ (normally distributed); $\odot$, $(S-\gamma S_1)/(1-\gamma)$ the residual component.)

component of larger variance and it is found to be Gaussian within the experimental uncertainty. Assuming this component to be normally distributed, it is possible to determine its variance and its relative duration from the measurements at large deviations and to obtain the other residual component by subtracting the inferred intense component from the measured probabilities. The steps in the analysis are indicated on the diagrams and the results of the process, which has been applied to all the measured distributions of $\partial\theta/\partial x$, are summarized in figures 15 to 17. These show the variation with height of the relative duration and intensity of the intense ('active') component and of the distribution of the residual component.

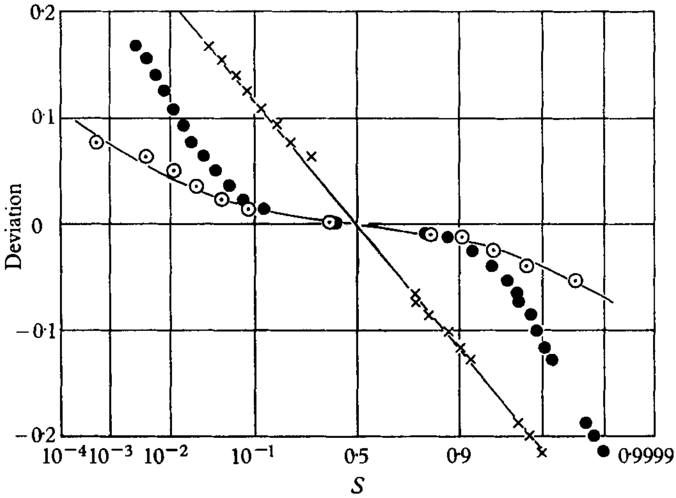

FIGURE 14. Statistical distribution of temperature gradient, $\partial\theta/\partial x$, at $z/z_0 = 22{\cdot}6$ on a Gaussian probability plot. ($\bullet$, original measurements; $\times$, $S/\gamma \to S_1$ (normally distributed); $\odot$, $(S-\gamma S_1)/(1-\gamma)$ the residual component.)

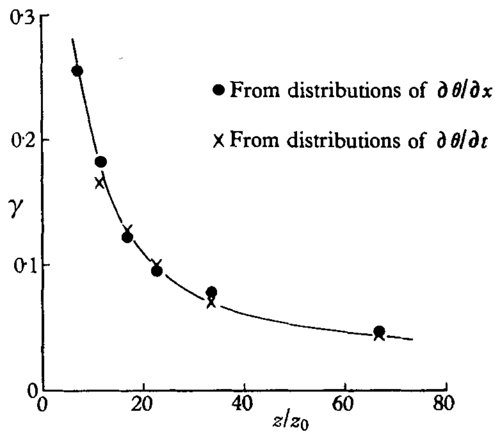

FIGURE 15. Variation with z/z_0 of the relative duration, γ, of periods of activity.

It is not possible to analyse the distributions of $\partial\theta/\partial t$ in the same way, as these distribution functions do not behave normally at large deviations, approaching zero or unity less rapidly than a normal distribution (figure 18). If the local Reynolds and Péclet numbers of the turbulent motion are large, the time rate of change of temperature is almost entirely due to convection of temperature gradients by the fluid velocities past the point of observation. In an appendix it is shown that the convection of a normally distributed field of temperature gradient by a normally distributed and uncorrelated field of velocity leads to rates of change of temperature with a distribution function of the form

$$P_c(x) = \tfrac{1}{2}|x|\, H_1^{(1)}(i\,|x|), \tag{9.1}$$

where $H_1^{(1)}(ix)$ is the Bessel function of imaginary argument tabulated by Jahnke & Emde (1933). For large positive values of x,

$$P_c(x) \sim \sqrt{\frac{x}{2\pi}}\, e^{-x}\left(1+\frac{3}{8x}-\frac{15}{128x^2}+\ldots\right). \tag{9.2}$$

The measurements are of the integrated distribution function, corresponding with

$$S_c(x) \sim \sqrt{\frac{x}{2\pi}}\, e^{-x}\left(1+\frac{7}{8x}-\frac{71}{128x^2}+\ldots\right), \tag{9.3}$$

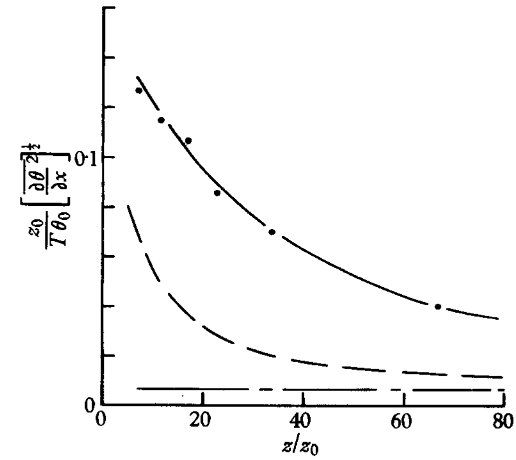

FIGURE 16. Variation with z/z_0 of the root-mean-square of the active component of $\partial\theta/\partial x$. (The dashed line indicates the directly observed values of $[(\partial\theta/\partial x)^2]^{\frac{1}{2}}$ given in figure 7, and the horizontal line the intensity of the quiescent component.)

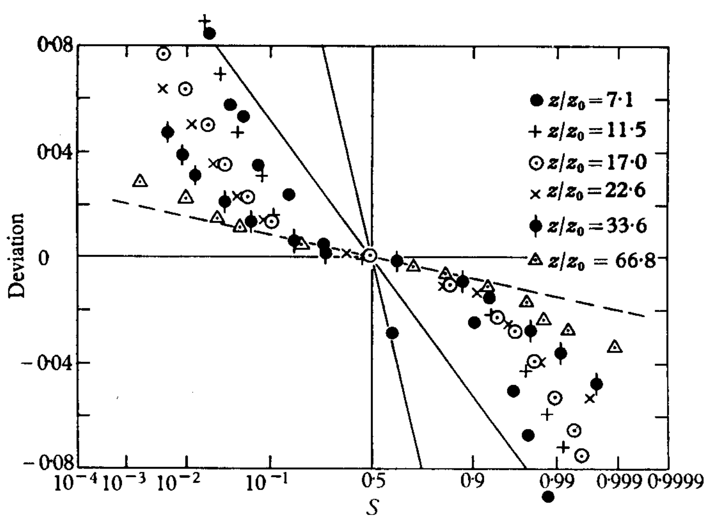

FIGURE 17. Statistical distributions of the residual components of $\partial\theta/\partial x$, i.e. what is left after subtracting the active component and renormalizing.

and the observations of the distributions of $\partial\theta/\partial t$ for large deviations are in much better accord with this type of asymptotic behaviour than the behaviour of a normal distribution.

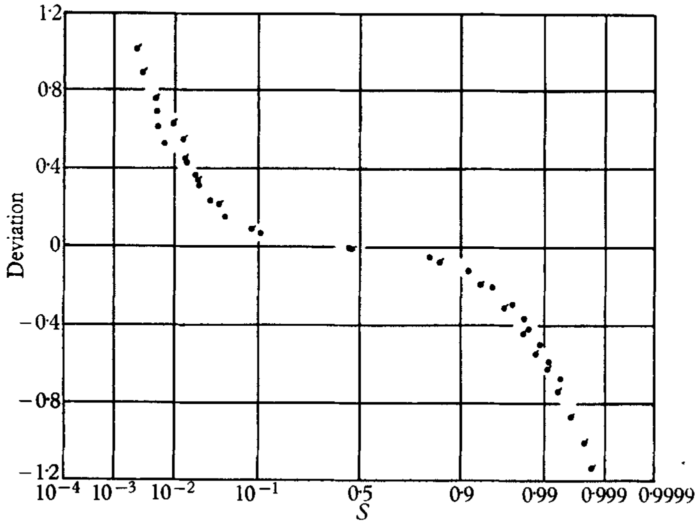

FIGURE 18. Gaussian probability plot of the statistical distribution of $\partial\theta/\partial t$ for $z/z_0 = 22{\cdot}6$.

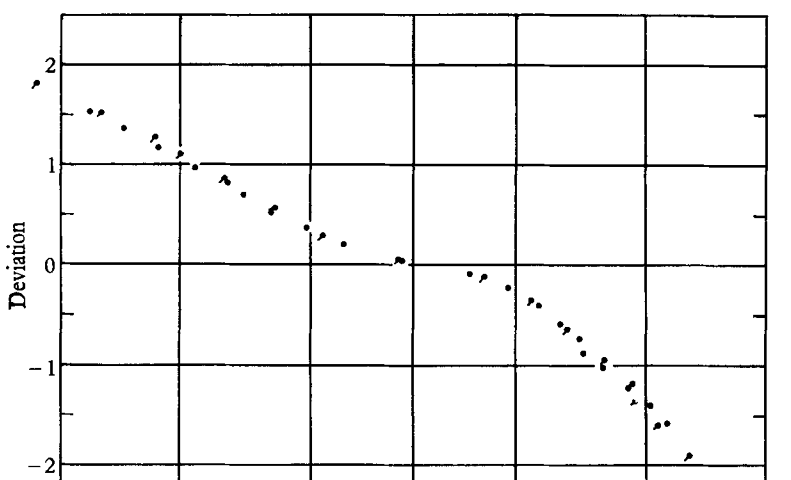

FIGURE 19. Statistical distribution of $\partial\theta/\partial t$ for $z/z_0 = 7{\cdot}1$ on a 'convection' probability plot.

The measured distributions of $\partial\theta/\partial t$ have been analysed into two components, the more intense one being distributed according to the 'convection' distribution (9.1) and having an intensity and relative duration determined by the observed probabilities of large deviations. In figures 19 and 20, two measured distributions have been plotted using the analogue of the Gaussian probability plot (i.e. one on which a distribution of the form (9.1) appears as a straight line), and the results of the analysis are shown in figures 15, 21 and 22 and in table 3. At the smaller values of z/z_0, the distributions of $\partial\theta/\partial t$ become asymmetrical and,

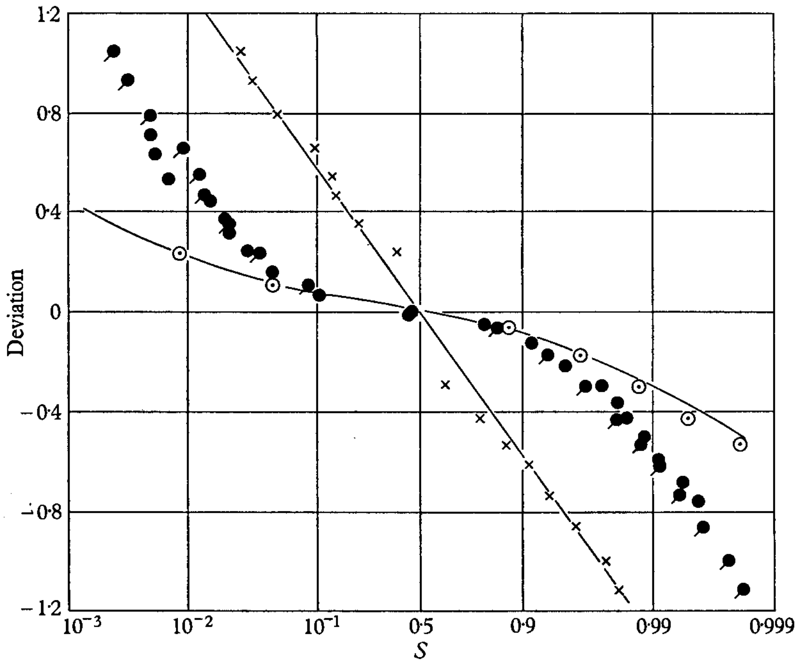

FIGURE 20. Statistical distribution of $\partial\theta/\partial t$ for $z/z_0 = 22{\cdot}6$ on a 'convection' probability plot. ($\bullet$, original measurements; $\times$, $S/\gamma \to S_1$ (normally distributed); $\odot$, $(S-\gamma S_1)/(1-\gamma)$ the residual component).

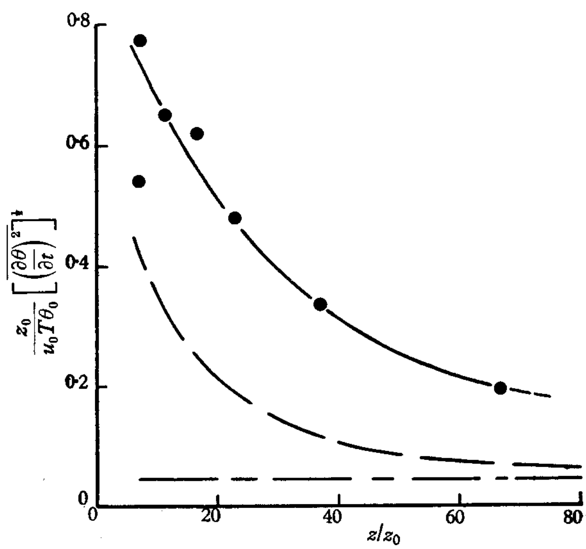

FIGURE 21. Variation with z/z_0 of the root-mean-square of the active component of $\partial\theta/\partial t$. (The dashed line indicates the directly observed values of $[\overline{(\partial\theta/\partial t)^2}]^{\frac{1}{2}}$ given in figure 9, and the horizontal line the intensity of the quiescent component.)

although they preserve the same asymptotic behaviour for both large positive and large negative deviations, it is not possible to analyse the distribution for $z/z_0 = 7{\cdot}1$ and barely possible for $z/z_0 = 11{\cdot}5$. The sense of the asymmetry is that large negative values of $\partial\theta/\partial t$ are more probable than large positive ones.

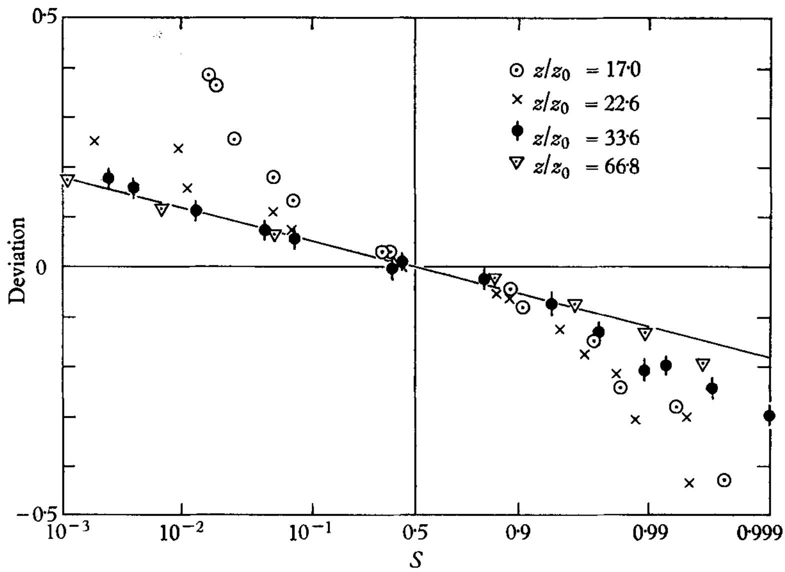

FIGURE 22. Statistical distributions of the residual components of $\partial\theta/\partial t$, i.e. what is left after subtracting the active component and renormalizing).

Quantity		7·1	11·5	17·0	22·6	33·6	66·8
$\partial\theta/\partial x$	$\gamma =$	0·255	0·182	0·121	0·096	0·078	0·047
	Variance	0·127	0·115	0·107	0·086	0·070	$0{\cdot}040_5$
$\partial\theta/\partial t$	$\gamma =$	—	0·165	0·128	0·099	0·072	0·046
	Variance	—	0·65	0·621	0·483	0·368	0·197

TABLE 3. Properties of the active components of $\partial\theta/\partial x$ and $\partial\theta/\partial t$.

A comparison of the two sets of distributions shows that: (i) The duration factors inferred from the distributions of $\partial\theta/\partial x$ and $\partial\theta/\partial t$ are consistent and decrease rapidly with increase of height. It is very likely that the 'active' components of $\partial\theta/\partial x$ and $\partial\theta/\partial t$ are consequences of the same 'active' periods.

(ii) The intensities of the active components also decrease rapidly with distance from the surface (figures 16, 21).

(iii) The distribution of the residual components is independent of height except for extreme values of the deviation with low probability of occurrence (figures 17, 22). The departures from normal and 'convection' distributions are least for the greatest heights and may be due to transition zones between the active and quiescent regions in the flow. The variances corresponding to the invariant parts of the residual components are respectively 0·0064 for $\partial\theta/\partial x$ and 0·045 for $\partial\theta/\partial t$, both in the appropriate non-dimensional units.

10. Comparison with theoretical predictions and with measurements in the atmosphere

Before discussing possible structures of the convection that would be consistent with the observations, it is convenient to consider the extent to which the several theories are confirmed by experiment. The primitive theory is that there exists a surface layer within which the temperature structure is uniquely determined by the heat flux, the gravitational acceleration and the molecular conductivity and viscosity. If this is true, measurements for all heat transfers should coincide if expressed in terms of the appropriate scales of temperature, length and velocity. This prediction is well confirmed by these and the previous measurements, except within the conduction layer and at considerable distances from the surface.* Deviations at considerable distances are believed to be caused by external air-currents and diminish as the intensity of the convection increases, while the deviations within the conduction layer are caused by the variation with temperature of thermometric conductivity and kinematic viscosity. A rough argument to establish the magnitude of the last effect is as follows. Within the conduction layer, eddying motions do not develop and the motion is in a critical state of stability. Just outside the layer, the fluid velocities are of order u_0 and the temperature differences across the layer are of order $T_1\theta_0$, so that the stability criterion is of the form

$$fn\left[\frac{u_0 z_c}{\nu_c},\quad \frac{g\theta_0 z_c^3}{\nu_c \kappa_c}\right] = 1, \tag{10.1}$$

where z_c is the layer thickness, and ν_c, k_c are mean values of the kinematic viscosity and thermometric conductivity in the layer. If u_0 and θ_0 are both set by the convection in the cooler fluid outside the layer, it is likely that the layer thickness varies at some rate intermediate between proportionality to ν_c ($u_0 z_c/\nu_c$ = constant) and proportionality to the $\frac{2}{3}$-power of ν_c ($g\theta_0 z_c^3(\nu_c\kappa_c)^{-1}$ = constant). Altering the horizontal scales of figures 4 and 6 in either of these ways removes the greater part of the variation with temperature loading.

The measurements of mean temperature and, much more definitely, the measurements of temperature fluctuations are not reconcilable with the similarity theory of convection and there can be little doubt that the theory does not apply to convection in the laboratory. It remains to account for the opposite conclusion drawn from measurements in the atmosphere, and here recent work by Webb (1958) has made it clear that wind-shear had an effect on the earlier measurements of Swinbank (Priestley 1954, 1955, 1956). Webb's measurements show that three regions may be distinguished in the earth's boundary layer with upward transfer of heat: (*a*) near the ground, a region of forced convection with temperature gradient inversely proportional to height; (*b*) farther up, a region of apparently free convection with temperature gradient varying as the $-\frac{4}{3}$-power of height and independent of wind-shear; and (*c*) an upper region of imperceptible temperature gradient. The boundaries between these regions may be expressed by critical

* It is unlikely that the fluctuations during periods of quiescence are unaffected by the distant boundaries, but their contribution to the total intensities is too small to falsify the prediction (see § 11).

values of the local Richardson number. The earlier measurements did not extend above the intermediate region but the existence of a third region shows that, although the temperature gradient may be independent of wind-shear in the intermediate region, this is only true for a limited range of wind-shear and a sufficient reduction in wind-shear will lead to transition to another type of convection, presumably similar to the convection over a flat surface without wind.

The imposition of a horizontal shearing motion on homogeneous natural convection may have several effects, but two obvious ones are a transfer of energy from the mean flow to the turbulent motion and a restriction on the freedom of vertical movement of the fluid. The first tends to increase turbulent velocities while the second restricts the vertical scale of the convective movements, and it is not obvious whether the presence of a shearing motion will increase or decrease the temperature gradient for a given heat flux. Webb's measurements show that the first effect of a moderate wind-shear is to increase the temperature gradient but that a condition is reached in which further increase of wind-shear is at first without effect on the temperature gradient and then decreases it as the contribution to the turbulent energy by working against Reynolds stresses becomes dominant. Our hypothesis is that this is caused by an initial decrease in vertical scale of the convective motion which is later offset by the increase in turbulent intensity arising from the transfer of energy from the mean flow.

Experimental support of this hypothesis could only be obtained from detailed measurements of the flow, but it is interesting that the present measurements indicate length scales much larger (relative to height above the surface) than occur in forced convection through a boundary layer. The destruction of temperature fluctuations by conductivity depends on a 'cascade' process of intensification of temperature gradients by the turbulent motion that, like the closely analogous process of turbulent energy dissipation proceeds at a rate independent of the viscosity and conductivity of the fluid if the Reynolds number of the motion is large. The rate of destruction of $\frac{1}{2}\overline{\theta^2}$ is then

$$3\kappa\overline{\left(\frac{\partial\theta}{\partial x}\right)^2} = c'\frac{u_s}{l_s}\overline{\theta^2}, \tag{10.2}$$

where u_s, l_s are scales of velocity and length characteristic of the turbulent motion. This relation is analogous to the established form for the energy dissipation,

$$\epsilon = -\nu\overline{u_i\nabla^2 u_i} = cu_s^3/l_s. \tag{10.3}$$

Measurements by Kistler, O'Brien & Corrsin (1956) of the decay of temperature fluctuations suggest that

$$3\kappa\overline{\left(\frac{\partial\theta}{\partial x}\right)^2} = \tfrac{1}{3}c'\frac{(\overline{w^2})^{\frac{1}{2}}}{L}\overline{\theta^2}, \tag{10.4}$$

where $c' \doteqdot 0{\cdot}8$ and L is the integral scale of the turbulent motion. Making use of the measurements of $\overline{\theta^2}$ and $\overline{\left(\frac{\partial\theta}{\partial x}\right)^2}$ as expressed by equations (6.7) and (7.3), it follows that

$$\frac{L}{z} = 2{\cdot}0c'\left(\frac{z}{z_0}\right)^{-0{\cdot}1}\frac{(\overline{w^2})^{\frac{1}{2}}(\overline{\theta^2})^{\frac{1}{2}}}{\overline{\theta w}}, \tag{10.5}$$

and that $L/z \doteqdot 2$ for the larger values of z/z_0. In forced convection through a turbulent boundary layer, $L/z \doteqdot 0{\cdot}4$, relatively much smaller. Obviously, the presence of the shearing motion exercises a powerful restriction on the scale of the motion.

The measurements support the Malkus theory of turbulence in that the observed distribution of mean temperature well outside the conduction layer is described by

$$\theta_0^{-1}\log\frac{T}{T_a} = 2{\cdot}6\left(\frac{z}{z_0}\right)^{-1} + 0{\cdot}06 \tag{10.6}$$

within the experimental error. It is interesting that the multiplicative constant, 2·6, based on these more extensive measurements is nearer to the theoretical value,* 2·0, than that based on the earlier measurements over a more limited range of z/z_0, and that the magnitude of the additive constant, 0·06, might have been inferred from a study of the distributions of temperature fluctuations (see § 11).

11. Physical description of the flow

Possibly the most striking feature of the observations is the existence of two modes of temperature fluctuation, but no hint of it is to be found in either theory of the convection and it must be a part of the physical mechanism of the convection that underlies the generalizations of the theory. If the temperature at a fixed point has two distinct modes of fluctuations, the space occupied by the fluid at any instant must be divided by comparatively well-defined bounding surfaces into corresponding regions of 'activity' and 'quiescence', and the properties of the modes are also those of the regions. Within the active regions temperature fluctuations are large and the mean temperature is high, while within the quiescent regions the fluctuations are much smaller and the mean temperature only slightly above the reference temperature. The proportion of the active mode in a period of observation decreases fairly rapidly with increase of height and the average time interval between successive periods of activity increases. As the fluctuations are, very nearly, statistically homogeneous over horizontal planes, these observations mean that both the number and the total area of the sections of active regions by a horizontal plane decrease as the height of the plane increases and this suggests that the source of active regions may be warm, buoyant air-currents rising from the conduction layer.

If rising currents of this sort exist, they must be very different from the convection plumes that rise from localized heat sources in still air because in this arrangement all the air is in vigorous turbulent motion. This could be inferred by observing that, far from the surface, active regions are too few to transport much of the heat flux and that the major part is necessarily carried by turbulent

* This 'theoretical' value would probably not be obtained from a strict application of the theory, as it depends on an analysis in which the orthogonal functions used to represent the fields of velocity and temperature satisfy some but not all of the boundary conditions (Malkus 1954*b*). It is likely that use of the correct orthogonal functions would lead to a somewhat larger multiplicative constant. A similar effect would arise from a subsequent modification of the theory (Malkus & Veronis 1958).

movements in the quiescent regions, but it is also possible to estimate the mean square of the turbulent velocity in each kind of region from the measured distributions of $\partial\theta/\partial x$ and $\partial\theta/\partial t$. If the Reynolds and Péclet numbers of a turbulent motion are large and if the fluctuations of velocity and temperature gradient are statistically independent, it may be shown that (see the Appendix)

$$\overline{\left(\frac{\partial\theta}{\partial t}\right)^2} = \overline{q^2}\,\overline{\left(\frac{\partial\theta}{\partial x}\right)^2}. \tag{11.1}$$

The first condition is satisfied for large enough values of z/z_0 but the second is not satisfied if the mean values extend over the whole time of observation without distinction between active and quiescent periods. Within regions of one kind, the distributions of $\partial\theta/\partial x$ and $\partial\theta/\partial t$ have, very nearly, the forms expected if both conditions were satisfied and if the distributions of velocity and temperature gradient were normal. Assuming this, the mean square of the turbulent velocity in each kind of region can be estimated, with fair accuracy for the active regions but with considerable uncertainty for the quiescent regions, by using the results of the analysis of the statistical distributions (table 3). In the active regions, the root-mean-square of the velocity is about $5{\cdot}5u_0$, almost independent of height but with a tendency to decrease slowly as z/z_0 increases. For the quiescent regions, variances computed from the central part of the observed distributions give a root-mean-square velocity of $7u_0$, but the uncertainty is considerable and this value may be too small. In any event, the turbulent velocities are somewhat greater in the 'quiescent' periods than they are in the 'active' periods.

It appears probable that active regions are formed by more or less localized emission of heat from the conduction layer, most likely in the neighbourhood of points or lines of flow 'separation' where the horizontal velocity just outside the conduction layer happens to be small or zero. The hot air rising from such a place will mix rapidly with the surrounding turbulent air, but in these circumstances the distinction between air that has received heat from the heat source and air that has not remains fairly sharp as is shown by observations of the spread of heat through a turbulent boundary layer (Batchelor 1954; Johnson 1955). The presence and persistence of 'up-draught sites' has been described by Malkus (1954*a*) who observed the motion of suspended particles in acetone and water, and these may be identified with the hypothetical heat sources.* The persistence of the sources is confirmed in these experiments by the comparatively long duration of the active periods (of order 10 sec compared with a scale time (z_0/u_0 of about 0·5 sec), and a possible explanation is that, once a site is established, it attracts to itself air heated by passage through the conduction layer which adds to the strength and stability of the up-draught.

The distribution of mean temperature is determined by the extent to which these up-draughts penetrate the cool 'quiescent' air, which is dependent on the initial cross-section and strength of the up-draught, both closely related to the thickness of the conduction layer. It is probably in this way that the scale length

* The existence of similar up-draughts in the atmosphere with light winds and uniform surface heating is well recognized, but they are often identified with convection plumes from isolated sources of heat.

z_0 enters into the temperature distribution far from the conduction layer and that the viscosity and conductivity of the fluid can affect the convection at those large values of z/z_0, for which their direct effects are negligible. Natural convection of this type will occur over a rough horizontal surface and in the outer part of a horizontal boundary layer of sufficient thickness, and then viscosity and conductivity are not expected to have any direct influence on the convection. If this view of the way in which molecular transfer influences the convection over a smooth plane is correct, the distribution of mean temperature will be described by the same equation (10.6) but with a value of the scale length related to the scale of the surface roughness or, for the boundary layer, to the horizontal scale of the temperature fluctuations at the level where natural convection becomes dominant.* In these circumstances, the convection depends on the quantities z_0, g and Q, and the scales of velocity and logarithmic temperature are

$$\left.\begin{aligned} u_0 &= (gQz_0)^{\frac{1}{3}}, \\ \theta_0 &= Q^{\frac{2}{3}}(gz_0)^{-\frac{1}{3}}. \end{aligned}\right\} \tag{11.2}$$

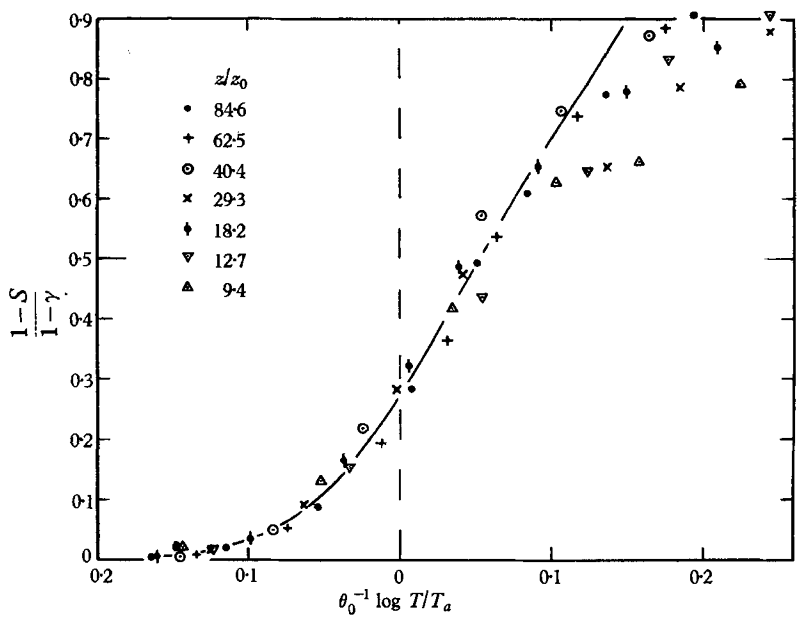

FIGURE 23. Statistical distributions of instantaneous temperature, plotted as $(1-S)/(1-\gamma)$ to show invariance of quiescent components.

Although the hot 'active' regions are certainly characteristic of the surface layer and would be expected to appear whatever the dimensions of the complete convection box, it is most unlikely that the temperature and velocity structure of the cool 'quiescent' regions is independent of these dimensions. It has been pointed out that the contributions of the 'quiescent' regions to the distributions of temperature and temperature gradient are nearly independent of height

* A very deep boundary layer with a constant Reynolds stress, τ_0, and a constant upward flux of total heat has a characteristic length $\tau_0^{\frac{3}{2}}(gQ)^{-1}$ (Ellison 1957) and the appropriate value of the scale length z_0 will be a multiple of this.

(allowing for the variation of duration factor), and this indicates that the temperature structure in these regions is characteristic of the whole flow and not just of the surface layer. The evidence for this statement is contained in figures 17 and 22, and in figure 23 which shows the statistical distributions of instantaneous temperature at various heights (previously given in figure 11) replotted in terms of this hypothesis using the duration factors inferred from analysis of the distributions of $\partial\theta/\partial x$ and $\partial\theta/\partial t$. The identity of form, particularly at the lower temperatures, and the apparent asymptotic approach to a universal form suggest strongly an essential homogeneity of temperature fluctuations in the quiescent regions. If the fluctuations in the quiescent regions are characteristic of the whole flow, it will not be possible to retain strict dimensional similarity but the contribution of these regions to the measured quantities is so small that a lack of dimensional similarity would be impossible to detect in these experiments.

It may be noticed that the medians of all the distributions in figure 23 lies very close to $\theta_0^{-1}\log(T+\theta)/T_a = 0{\cdot}055$, and, since there is no indication that the asymptotic statistical distribution of temperature is strongly asymmetric, the mean value at great heights must also lie near 0·055. This observation provides a meaning for the additive constant, 0·06, in the equation (10.6) which describes the variation of mean temperature as the mean temperature of the air far from the surface layer.

Although it is difficult to compare the measurements of velocity fluctuations in turbulent flows of constant density with these measurements of temperature fluctuations, there is little doubt that the turbulent motion over a heated horizontal surface differs in kind from the turbulent motion in a boundary layer. At first sight, this lack of resemblance is very surprising and its confirmation is probably the most convincing argument in favour of the theory of turbulence proposed by Malkus which, with the same set of assumptions, produces all the accepted results for wall turbulence (Malkus 1956) and makes predictions for thermal turbulence that are consistent with experiment. In the theory, the differences appear to arise from a difference in nature of the corresponding problems of laminar stability but a direct reason for the inapplicability of similarity considerations to thermal turbulence can be found by considering the balance of turbulent kinetic energy that would occur if the similarity theory were valid. To the usual approximation, the equation for the kinetic energy is

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\overline{pw}+\tfrac{1}{2}\overline{q^2w}\right) = \frac{g}{T}\overline{\theta w}-\epsilon, \tag{11.3}$$

where ϵ is the local rate of conversion of kinetic energy to heat. The similarity theory predicts that

$$(\overline{w^2})^{\frac{1}{2}} = a(gQz)^{\frac{1}{3}}, \quad \frac{1}{\rho}\overline{pw}+\tfrac{1}{2}\overline{q^2w} = -\beta(\overline{w^2})^{\frac{3}{2}}, \quad \epsilon = \frac{(\overline{w^2})^{\frac{3}{2}}}{L_\epsilon}, \tag{11.4}$$

where L_ϵ is proportional to z and is nearly the integral scale of the turbulent motion. Substitution of these relations in the energy equation leads to an expression for a,

$$a = \left(\frac{z}{L_\epsilon}-\beta\right)^{-\frac{1}{3}}. \tag{11.5}$$

This quantity determines the general level of turbulent intensity and is essentially positive, while β, which describes the transport of energy from one part of the flow to another is positive if energy transport takes place down intensity gradients. Measurements in constant density flows with gradients of turbulent intensity show that z/L_ϵ and β are very similar in magnitude (e.g. Townsend 1951), and it is at least possible that β is greater than z/L_ϵ in which event equation (11.5) has no meaning except that transport of turbulent energy from one part of the flow to another prevents the establishment of a motion described by equations (11.4). If transfer of turbulent energy dominates the balance of kinetic energy, it is unlikely that similarity considerations, based on the notion that the surface layer is independent of both the conduction layer and the outer flow, are ever applicable. This kind of difficulty does not arise in the treatment of wall turbulence in which the gradient of turbulence intensity and the net energy flux are both negligible.

A further point of difference has already been mentioned, the presence of shearing motion tending to restrict the effective scale of turbulent movements, which is greater in natural convection without shear.

12. Production and destruction of temperature fluctuations

A direct consequence of the heat equation and of the symmetry of the flow is that

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\overline{\theta^2}) + \left[QT + \kappa\frac{\partial T}{\partial z}\right]\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}(\tfrac{1}{2}\overline{\theta^2 w}) = \kappa\frac{\partial^2}{\partial z^2}(\tfrac{1}{2}\overline{\theta^2}) - \kappa\left[2\overline{\left(\frac{\partial\theta}{\partial x}\right)^2} + \overline{\left(\frac{\partial\theta}{\partial z}\right)^2}\right]. \quad (12.1)$$

The first term, the time rate of change of $\tfrac{1}{2}\overline{\theta^2}$, is zero for steady convection but has been included to make clear the meaning of the equation as the balance between the production, transport and destruction of temperature fluctuations. The remaining terms represent production of temperature fluctuations by interaction between the turbulent motion and the gradient of mean temperature, the net effect of convection of temperature fluctuations by turbulent movements, the net effect of molecular diffusion of temperature fluctuations, and the destruction of fluctuations by conduction.

Well outside the conduction layer, it is likely that

$$\overline{\left(\frac{\partial\theta}{\partial x}\right)^2} = \overline{\left(\frac{\partial\theta}{\partial z}\right)^2}, \quad (12.2)$$

and then all the terms of equation (12.1) may be calculated from the results of this work, excepting the one describing turbulent convection which may be obtained by difference. For the region $z/z_0 > 10$ then, equation (12.1) may be written in the non-dimensional form

$$-2{\cdot}6\left(\frac{z}{z_0}\right)^{-2} + \frac{z_0}{u_0\theta_0^2T^2}\frac{\partial}{\partial z}(\tfrac{1}{2}\overline{\theta^2 w}) = -0{\cdot}27\left(\frac{z}{z_0}\right)^{-1\cdot5}, \quad (12.3)$$

where use has been made of equations (6.2) and (7.3) to represent the observed variations of mean temperature and mean square temperature gradient, and two terms, $\kappa\left(\frac{\partial T}{\partial z}\right)^2$ and $\kappa\frac{\partial^2}{\partial z^2}(\tfrac{1}{2}\overline{\theta^2})$, are negligible and have been omitted. The integral of this equation is

$$2{\cdot}6\left(\frac{z}{z_0}\right)^{-1}+\frac{\overline{\theta^2 w}}{2u_0\theta_0^2 T^2} = 0{\cdot}54\left(\frac{z}{z_0}\right)^{-0\cdot5}+a_0, \tag{12.4}$$

where a_0 is a constant of integration. If the power-laws used to describe the variations of mean temperature and mean square temperature gradient were valid for all large values of z/z_0, the constant would necessarily be zero, but the double structure of the convection and the identification of the quiescent regions with the flow 'at infinity' make it extremely unlikely that simple power laws will be valid except as approximations within the range of observation. In principle, the constant could be calculated by integrating equation (12.1) from the surface to a point in the range of validity of equation (12.4), but the accuracy of this procedure would be very low since most of the production and destruction of fluctuations takes place in the layer z/z_0 less than five, and small inaccuracies in the estimation of, say, rate of destruction would lead to an uncertainty in a_0 large compared with terms in equation (12.4).

The main conclusion from this analysis is that the production of fluctuations tends to exceed the destruction over most of the fully turbulent region, and it follows that convection of temperature fluctuations by turbulent movements is of importance.

Appendix: The relation between rate of change of temperature and temperature gradient in turbulent flow

In flows of nearly constant density, the temperature fluctuation satisfies the equation

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t}+\mathbf{u}.\nabla\theta = \kappa\nabla^2\theta. \tag{1}$$

If the Reynolds number of the turbulent motion is large, the substantial rate of change of temperature, $D\theta/Dt$, is usually small compared with either $\partial\theta/\partial t$ or $\mathbf{u}.\nabla\theta$ and the rate of change of temperature at a point fixed in space is related to the local velocity and temperature gradient by

$$\frac{\partial\theta}{\partial t}+\mathbf{u}.\nabla\theta = 0. \tag{2}$$

The justification for this approximation is that the greater part of the temperature gradient is derived from Fourier components of the temperature field with wave-numbers near $(\epsilon/\nu^3)^{\frac{1}{4}}$ (ϵ is the rate of viscous dissipation of turbulent energy),* and so the time rate of change of temperature by molecular conduction is of order $\kappa(\epsilon/\nu^3)^{\frac{1}{2}}\theta_d$, where θ_d is the 'amplitude' of the Fourier components which contribute to the temperature gradient. This rate should be compared with the rate of change due to convection of the temperature field past the point of observation

* This discussion is only valid for fluids for which the Prandtl number, ν/κ, is near one. For a detailed account of the spectrum of the temperature fluctuations, see Batchelor (1959).

which is of order $u_0\theta_d(\epsilon/\nu^3)^{\frac{1}{4}}$, where u_0 is the root-mean-square of the turbulent velocity. It follows that the order of magnitude of $\mathbf{u}.\nabla\theta$ is greater than that of $D\theta/Dt = \kappa\nabla^2\theta$ in the ratio $u_0\kappa^{-1}(\epsilon/\nu^3)^{-\frac{1}{4}}$. It is well known that $\epsilon = u_0^3/L_\epsilon$, where L_ϵ is nearly equal to the integral scale of the turbulent motion, and so the ratio is nearly $(u_0L_\epsilon/\nu)^{\frac{3}{4}}$ which is large if the Reynolds number of the motion is large.

Using this approximation, we may derive the relation between the probability distribution functions of $\mathbf{u}$, of $\nabla\theta$ and of $\partial\theta/\partial t$. Consistently with the assumption of large Reynolds number, we assume that the turbulent velocity and the temperature gradient are statistically independent and that the temperature gradients are isotropically distributed.* Then the distribution function for $\partial\theta/\partial t$ is

$$P_1\left(\frac{\partial\theta}{\partial t}\right) = \int_0^\infty P_2(u)\,P_3\left(-\frac{\dot{\theta}}{u}\right)\frac{du}{u}, \tag{3}$$

where $P_2(u)\,du$ is the probability that the turbulent velocity should lie between u and $u+du$, and $P_3(\theta_x)$ is the probability that the component of the temperature gradient in a particular direction (actually in the direction of $\mathbf{u}$) should have the value θ_x. It is a simple consequence of equation (3) that the product of the nth-order moments of the distributions of $|\mathbf{u}|$ and θ_x equals the nth-order moment of the distribution of $\partial\theta/\partial t$. In particular, the second moments are related by

$$\overline{\left(\frac{\partial\theta}{\partial t}\right)^2} = \overline{|\mathbf{u}|^2}\,\overline{\left(\frac{\partial\theta}{\partial r}\right)^2} = \overline{q^2}\,\overline{\left(\frac{\partial\theta}{\partial x}\right)^2}, \tag{4}$$

a result obtainable directly from equation (2).

The product rule which relates the moments of the three distributions implies that one at least of $\mathbf{u}$, $\nabla\theta$ and $\partial\theta/\partial t$ is not normally distributed, and the distribution of $\partial\theta/\partial t$ is the most likely to depart from normality. It is interesting to compute the distribution function for $\partial\theta/\partial t$, assuming that $\mathbf{u}$ and $\nabla\theta$ are normally distributed. In real flows, it is unlikely that these last two quantities have exactly normal distributions but the experimental evidence suggests that the departures from normality may not be large. We set

$$P_2(u) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} u^2\,e^{-\frac{1}{2}u^2}, \quad P_3(\theta_x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\theta_x^2}, \tag{5}$$

corresponding to normal distributions with variances $\sqrt{3}$ and 1 respectively. Then,

$$\begin{aligned} P_c(\dot{\theta}) &= \frac{1}{\pi}\int_0^\infty u\exp\left[-\frac{1}{2}\left(u^2+\frac{\dot{\theta}^2}{u^2}\right)\right]du \\ &= \frac{\dot{\theta}}{2\pi}\int_0^\infty \exp\left[-\tfrac{1}{2}\dot{\theta}(\lambda+\lambda^{-1})\right]d\lambda = -\tfrac{1}{2}\dot{\theta}H_1^{(1)}(i\dot{\theta}), \end{aligned} \tag{6}$$

* Statistical independence of turbulent velocity and temperature gradient requires that, if there exist regions or periods of unusually large velocity fluctuations, these are not associated either with unusually large or small fluctuations of temperature gradient. If they are, it may still be possible to use this analysis by confining attention to regions in which turbulent velocity and temperature gradient have a spatially uniform distribution of fluctuations.

in the notation used by Jahnke & Emde (1933) for the Bessel functions of imaginary argument. The probability distribution function specified by equation (6) is very different from a normal distribution as can be seen in figure 24.

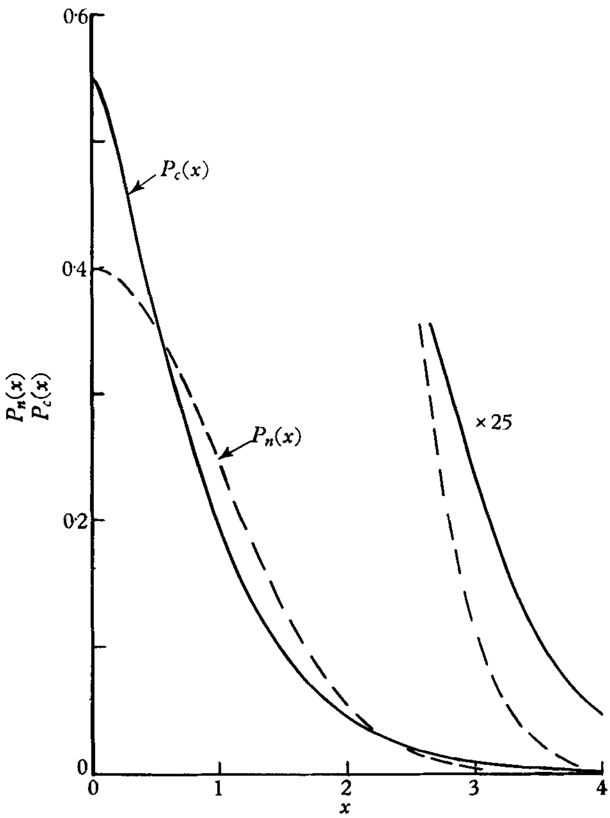

FIGURE 24. Comparison of normal Gaussian and 'convection' probability distribution functions of unit variance.

For large values of $\dot{\theta}$,

$$P_c(\dot{\theta}) = \left(\frac{\dot{\theta}}{2\pi}\right)^{\frac{1}{2}} e^{-\dot{\theta}}\left[1+\frac{3}{8\dot{\theta}}-\frac{15}{128\dot{\theta}^2}+\dots\right], \tag{7}$$

in contrast to the normal distribution function

$$P_n(x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}x^2}. \tag{8}$$

The experimental measurements are of the probabilities that $\partial\theta/\partial t$ and $\partial\theta/\partial x$ shall exceed set values, i.e.

$$S_1(\dot{\theta}) = \int_{\dot{\theta}}^{\infty} P_1(x)\,dx \quad \text{and} \quad S_3(\theta_x) = \int_{\theta_x}^{\infty} P_3(x)\,dx,$$

and comparison should be made with the integral of the distribution function defined by equation (6),

$$S_c(\dot{\theta}) = -\int_{\dot{\theta}}^{\infty} \frac{x}{2} H_1^{(1)}(ix)\,dx. \tag{9}$$

Values of $S_c(x)$ have been calculated, in part by numerical integration and in part using the asymptotic series,

$$S_c(x) = \left(\frac{x}{2\pi}\right)^{\frac{1}{2}} e^{-x}\left[1+\frac{7}{8x}-\frac{71}{128x^2}+\frac{765}{1024x^3}-\ldots\right] \tag{10}$$

with the results shown in table 4. For convenience, distributions similar to those defined by the functions $P_c(x)$ and $S_c(x)$ may be called 'convection' distributions. The most obvious difference from normal distributions is the relatively greater probability of occurrence of very large deviations.

x	$P_c(x)$	$S_c(x)$	x	$P_c(x)$	$S_c(x)$
0	0·3183	0·5000	2·5	—	0·0666
0·2	0·3040	0·4378	3·0	0·0384	0·0427
0·4	0·2782	0·3796	3·5	—	0·0274
0·6	0·2488	0·3269	4·0	0·0159	0·0174
0·8	0·2194	0·2800	4·5	—	0·0110
1·0	0·1916	0·2390	5·0	—	0·00695
1·2	0·1660	0·2032	5·5	—	0·00437
1·4	0·1430	0·1723	6·0	—	0·00274
1·6	0·1226	0·1457	6·5	—	0·00169
1·8	0·1046	0·1230	7·0	—	0·00102
2·0	0·0890	0·1036			

TABLE 4. 'Convection' distribution functions

REFERENCES

BATCHELOR, G. K. 1954 *Quart. J. Roy. Met. Soc.* **80**, 339.

BATCHELOR, G. K. 1959 *J. Fluid Mech.* **5**, 113.

ELLISON, T. H. 1957 *J. Fluid Mech.* **2**, 456.

JAHNKE, E. & EMDE, F. 1933 *Tables of Functions*. Teubner: Leipzig and Berlin.

JOHNSON, D. S. 1955 *Johns Hopkins Univ., Rep.* no. OSR TN 55–289.

KISTLER, A. L., O'BRIEN, V. & CORRSIN, S. 1956 *J. Aero. Sci.* **23**, 96.

LIEPMANN, H. W. 1952 *Z. angew. Math. Phys.* **3**, 321.

MALKUS, W. V. R. 1954*a* *Proc. Roy. Soc.* A, **225**, 185.

MALKUS, W. V. R. 1954*b* *Proc. Roy. Soc.* A, **225**, 196.

MALKUS, W. V. R. 1956 *J. Fluid Mech.* **2**, 521.

MALKUS, W. V. R. & VERONIS, G. 1958 *J. Fluid Mech.* **4**, 225.

PRIESTLEY, C. H. B. 1954 *Austr. J. Phys.* **7**, 176.

PRIESTLEY, C. H. B. 1955 *Quart. J. Roy. Met. Soc.* **81**, 139.

PRIESTLEY, C. H. B. 1956 *Proc. Roy. Soc.* A, **238**, 287.

RICE, S. O. 1944 *Bell. Tech. J.* **23**, 282.

RICE, S. O. 1945 *Bell. Tech. J.* **24**, 46.

THOMAS, D. B. 1956 Ph.D. dissertation: University of Cambridge.

THOMAS, D. B. & TOWNSEND, A. A. 1957 *J. Fluid Mech.* **2**, 473.

TOWNSEND, A. A. 1951 *Proc. Camb. Phil. Soc.* **47**, 375.

TOWNSEND, A. A. 1959 *J. Fluid Mech.* (to be published).

WEBB, E. K. 1958 *Quart. J. Roy. Met. Soc.* **84**, 118.